Homework Assignments

Bifurcations: Theory and Applications

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http://dynamics.mi.fu-berlin.de/lectures/

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Let us recall some results mentioned in the lecture.

Theorem: [Flow box theorem]
For $x \in X = \mathbb{R}^N$ and $f \in C^1(X, X)$, consider the ODE

$$\dot{x} = f(x).$$

Assume that $f(0) \neq 0$. Then there exists a local C^1 diffeomorphism Ψ near $x = 0$, such that $y = \Psi(x)$ satisfies

$$\dot{y} = e_1 \coloneqq \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Theorem: [Shoshitaishvili theorem]
For $x \in X = \mathbb{R}^N$ and $f \in C^1(X, X)$, consider the ODE

$$\dot{x} = f(x) = Ax + g(x),$$

where $g(0) = 0$ and $A \coloneqq D_x f(0)$. Then there exists a local C^0 homeomorphism Ψ near $x = 0$, such that $y \coloneqq \Psi(x)$ satisfies

$$\begin{aligned} \dot{y}^c &= h(y^c), \\ \dot{y}^h &= A^h y^h. \end{aligned}$$

The superscripts c and h respectively denote the projection of the dynamics on the center and hyperbolic parts of the spectral splitting given by the linearization A.

Problem 1: [Flow box theorem with parameters]
For $x \in X = \mathbb{R}^N$, and parameters $\lambda \in \Lambda = \mathbb{R}^k$, and $f \in C^1(\Lambda \times X, X)$, consider the ODE

$$\dot{x} = f(\lambda, x).$$

Use the standard flow box theorem quoted above to show that, for $|\lambda|$ small enough, there exists a λ-dependent local C^1 diffeomorphism $\Psi(\lambda, \cdot)$ near $x = 0$, such that $y = \Psi(\lambda, x)$ satisfies

$$\dot{y} = e_1.$$

Can you choose $\Psi \in C^1$?

Problem 2: [Grobman-Hartman theorem with parameters]
Use the Shoshitaishvili theorem quoted above to prove the following Grobman-Hartman theorem with parameters:
For $x \in X = \mathbb{R}^N$, and parameters $\lambda \in \Lambda = \mathbb{R}^k$, and $f \in C^1(\Lambda \times X, X)$, consider the ODE

$$\dot{x} = f(\lambda, x).$$

Here $f(0,0) = 0$ and $A := D_x f(0,0)$ is hyperbolic. Then, for $|\lambda|$ small enough, there exists a family of λ-dependent local homeomorphisms $\Psi(\lambda, \cdot)$ near $x = 0$, such that $y := \Psi(\lambda, x)$ satisfies

$$\dot{y} = Ay.$$

Can you choose $\Psi \in C^0$?

Problem 3: [Shoshitaishvili theorem with parameters]
Use the Shoshitaishvili theorem quoted above to prove the following Shoshitaishvili theorem with parameters:
For $x \in X = \mathbb{R}^N$, and parameters $\lambda \in \Lambda = \mathbb{R}^k$, and $f \in C^1(\Lambda \times X, X)$, consider the ODE

$$\dot{x} = f(\lambda, x).$$

Here $f(0,0) = 0$ and $A := D_x f(0,0)$. Then, for $|\lambda|$ small enough, there exists a family of λ-dependent local homeomorphisms $\Psi(\lambda, \cdot)$ near $x = 0$, such that $y := \Psi(\lambda, x)$ satisfies

$$\begin{aligned} \dot{y}^c &= h(\lambda, y^c), \\ \dot{y}^h &= A^h y^h. \end{aligned}$$

Can you choose $\Psi \in C^0$?

Problem 4: Consider the Arnol'd cat map $A : \mathbb{T}^2 \to \mathbb{T}^2$ on the torus $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ given by the $\mathrm{SL}(2,\mathbb{Z})$-matrix

$$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

(i) Show that A maps the lattice $\{(p/n, q/n) \in \mathbb{R}^2 \mid p, q \in \mathbb{Z}\}/\mathbb{Z}^2 \subset \mathbb{T}^2$ to itself, for any $n \in \mathbb{N}$.

(ii) Show that the periodic points of the cat map are a dense subset of $\mathbb{T}^2$.

(iii) Show that the (global) unstable manifold $W^u(0)$ of 0 is dense in $\mathbb{T}^2$.

(iv) Show that the homoclinic points to 0 are dense.

[Extra credit] Are these statements also true for any hyperbolic matrix $A \in \mathrm{SL}(2,\mathbb{Z})$, i.e. for $1 \notin |\mathrm{spec}\, A|$?